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Okulov, Valery; Fukumoto, Y.

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BOOK OF ABSTRACTS

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ASYMPTOTIC EXPANSIONS FOR MOTION OF A CURVED VORTEX FILAMENT TUBE WITH ELLIPTICALLY DEFORMED CORE

Y. Fukumoto¹ and V. L. Okulov^{2,3}

¹ Institute of Mathematics for Industry, Kyushu University, 819-0395 Fukuoka, Japan

E-mail: yasuhide@imi.kyushu-u.ac.jp

² Wind Energy Department, Technical University of Denmark, DK-2800 Lyngby, Denmark

³ Institute of Thermophysics, Siberian Branch of the Russian Academy of Sciences, 630090
Novosibirsk, Russia

E-mail: vaok@dtu.dk

Abstract. Three-dimensional motion of a slender vortex tube, embedded in an inviscid incompressible fluid, is investigated for the Euler equations. Using the method of matched asymptotic expansions in a small parameter ϵ , the ratio of core radius to curvature radius, the velocity of a vortex tube is derived to $O(\epsilon^3)$, whereby the influence of elliptical deformation of the core due to the self-induced strain is taken into account. Some detailed analysis is made of motion of a helical vortex tube.

Keywords: *vortex tube, Biot-Savart law, matched asymptotic expansions, helical vortex*

1. Introduction

At a large Reynolds number, vorticity has a tendency to accumulate in thin regions owing to the nonlinear effect. Vortex tubes endowed with curvature moves themselves. The local self-induced flow around the core comprises not only a uniform flow but also a straining field which deforms the core into an ellipse. Our purpose is to present a systematic method for calculating the effect of strained core on the traveling speed of a slender vortex tube, embedded in an inviscid incompressible fluid.

We employ the method of matched asymptotic expansions in a small parameter ϵ , the ratio of the core to the curvature radii, developed so far to $O(\epsilon)$. The local self-induced straining field makes its appearance at $O(\epsilon^2)$ and influences the traveling speed at the next order. We are thus required to extend asymptotic expansions to $O(\epsilon^3)$. Fukumoto & Moffatt (2000) carried through a substantial part of this program for an axisymmetric problem. Dyson's third-order velocity formula (Dyson 1893) for an inviscid vortex ring is recovered if specialized to a particular distribution of vorticity. We shall demonstrate that the restriction of axisymmetry can be lifted.

The outer solution is provided by the Biot-Savart law. To $O(\epsilon)$, the vorticity is shown to have the tangential component only. With this distribution, the vector potential for the velocity field. Use of the shift-operator technique systematically produces multiple-pole expansions. The detail of vorticity distribution is as yet unknown, and is determined, order by order, by the inner expansion and the matching procedure. This is supplemented by the contribution from transversal components of the vorticity which arises at $O(\epsilon^2)$. We point out that the previous theories did not concern the induced field due to di- and higher-poles. The inner solution is constructed by solving the Euler equations in the comoving coordinates, in the form of power series with respect to ϵ . An analysis tells that axial flow is assumed to be absent at leading order, but that the variation of curvature along the centerline and torsion give rise to pressure gradient along the filament at $O(\epsilon^2)$ which drives an axial flow at the same order. The third-order correction to the traveling speed is deduced by handling the dipole component in the flow in the orthogonal plane.

2. Setting of problem

In order to look into the flow field near the core, it is expedient to introduce local coordinates $(\tilde{x}, \tilde{y}, \xi)$, along with local cylindrical coordinates (r, φ, ξ) such that $\tilde{x} = r \cos \varphi$ and $\tilde{y} = r \sin \varphi$, moving with the filament. Here ξ is a parameter along the central curve \mathbf{X} of the vortex tube, defined so as to satisfy $\dot{\mathbf{X}}(\xi, t) \cdot \mathbf{t}(\xi, t) = 0$. Here a dot stands for a derivative in t with fixing ξ . Given a point \mathbf{x} sufficiently close to the core, there corresponds uniquely the nearest point $\mathbf{X}(\xi, t)$ on the centerline of filament. Then \mathbf{x} is expressed as

$$\mathbf{x} = \mathbf{X}(\xi, t) + r \cos \varphi \mathbf{n} + r \sin \varphi \mathbf{b}. \quad (1)$$

The coordinates (r, φ, ξ) are converted into orthogonal ones (r, θ, ξ) by adjusting the origin of angle as

$$\theta(\varphi, \xi, t) = \varphi - \int_{s_0}^{s(\xi, t)} \tau(s', t) ds', \quad (2)$$

where $s = s(\xi, t)$ is the arclength along the centerline [7].

We define the relative velocity $\mathbf{V} = (u, v, w)$ as functions of r, θ, ξ and t by

$$\mathbf{v} = \dot{\mathbf{X}}(\xi, t) + u \mathbf{e}_r + v \mathbf{e}_\theta + w \mathbf{t}, \quad (3)$$

where \mathbf{e}_r and \mathbf{e}_θ are the unit vectors in the radial and azimuthal directions respectively. The vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is calculated through

$$\begin{aligned} \boldsymbol{\omega} &= \omega_r \mathbf{e}_r + \omega_\theta \mathbf{e}_\theta + \zeta \mathbf{t} \\ &= \left\{ \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{1}{h_3} \frac{\partial v}{\partial \xi} + \frac{\eta}{h_3} \kappa w \sin \varphi - \frac{1}{h_3} \frac{\partial \dot{\mathbf{X}}}{\partial \xi} \cdot \mathbf{e}_\theta \right\} \mathbf{e}_r \\ &\quad + \left\{ -\frac{\partial w}{\partial r} + \frac{1}{h_3} \frac{\partial u}{\partial \xi} + \frac{\eta}{h_3} \kappa w \cos \varphi + \frac{1}{h_3} \frac{\partial \dot{\mathbf{X}}}{\partial \xi} \cdot \mathbf{e}_r \right\} \mathbf{e}_\theta \\ &\quad + \left\{ \frac{1}{r} \frac{\partial}{\partial r} (rv) - \frac{1}{r} \frac{\partial u}{\partial \theta} \right\} \mathbf{t}, \end{aligned} \quad (4)$$

where

$$\eta = \left| \frac{\partial \mathbf{X}}{\partial \xi} \right|, \quad h_3 = \eta (1 - \kappa r \cos \varphi). \quad (6)$$

We are concerned with a *quasi-steady* motion of a vortex filament. Suppose that the leading-order flow is circulatory motion with prescribed velocity field $v^{(0)}(r)$ as a function only of r . Consistently with the LIA, we may pose the following form for the perturbation solution in a power series in $\epsilon = \sigma_0/R_0$, the ratio of a typical core radius σ_0 to a typical curvature radius R_0 :

$$\begin{aligned} u &= \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} + \dots, \\ v &= v^{(0)}(r) + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} + \dots, \\ w &= \epsilon^2 w^{(2)} + \dots, \quad \dot{\mathbf{X}} = \dot{\mathbf{X}}^{(0)} + \epsilon^2 \dot{\mathbf{X}}^{(2)} + \dots. \end{aligned} \quad (7)$$

Inspection from (4) and (6) tells us that

$$\begin{aligned} \omega_r &= \epsilon^2 \omega_r^{(2)} + \dots, \quad \omega_\theta = \epsilon^2 \omega_\theta^{(2)} + \dots, \\ \zeta &= \zeta^{(0)}(r) + \epsilon \zeta^{(1)} + \epsilon^2 \zeta^{(2)} + \epsilon^3 \zeta^{(3)} + \dots. \end{aligned} \quad (8)$$

To integrate the Euler equations, it is advantageous to eliminate the pressure at the outset and to deal exclusively with vorticity and vector potential \mathbf{A} for the velocity: $\mathbf{v} = \nabla \times \mathbf{A}$. Introduce a Stokes stream-function

$$\psi(\mathbf{x}) = (1 - \kappa r \cos \varphi) \mathbf{A}(\mathbf{x}) \cdot \mathbf{t}(\xi) \quad (9)$$

$$= \psi^{(0)}(r) + \epsilon \psi^{(1)} + \epsilon^2 \psi^{(2)} + \epsilon^3 \psi^{(3)}. \quad (10)$$

3. Asymptotic development of Biot-Savart law

This section presents only a brief sketch of how to perform an asymptotic development, valid near the core, of the Biot-Savart law for $\mathbf{A}(\mathbf{x})$.

The vorticity is dominated by the tangential contribution ζ . We stipulate that $|\zeta|$ decays sufficiently rapidly to zero with distance r from the vortex centerline. The contribution \mathbf{A}_{\parallel} from ζ is

$$\mathbf{A}_{\parallel}(\mathbf{x}) = \frac{1}{4\pi} \iiint \frac{\zeta(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \mathbf{t}(s) (1 - \kappa \tilde{\mathbf{x}})}{|\mathbf{x} - \mathbf{X} - \tilde{\mathbf{x}}\mathbf{n} - \tilde{\mathbf{y}}\mathbf{b}|} d\tilde{\mathbf{x}} d\tilde{\mathbf{y}} ds. \quad (11)$$

Use of a shift-operator, being adapted from Dyson's technique [5], facilitates to rewrite (11) in a form amenable to a multi-pole expansion as

$$\begin{aligned} \mathbf{A}_{\parallel}(\mathbf{x}) &= \frac{1}{4\pi} \int ds \left\{ \iint d\tilde{\mathbf{x}} d\tilde{\mathbf{y}} \zeta(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) (1 - \kappa \tilde{\mathbf{x}}) \right. \\ &\quad \times \exp[-\tilde{\mathbf{x}}(\mathbf{n} \cdot \nabla) - \tilde{\mathbf{y}}(\mathbf{b} \cdot \nabla)] \left. \right\} \frac{\mathbf{t}(s)}{|\mathbf{x} - \mathbf{X}(s)|} \\ &= \frac{1}{4\pi} \int ds \left\{ \iint d\tilde{\mathbf{x}} d\tilde{\mathbf{y}} \zeta \left(1 - \kappa \tilde{\mathbf{x}} - \tilde{\mathbf{x}}(\mathbf{n} \cdot \nabla) - \tilde{\mathbf{y}}(\mathbf{b} \cdot \nabla) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} [\tilde{\mathbf{x}}^2 (\mathbf{n} \cdot \nabla)^2 + 2 \tilde{\mathbf{x}} \tilde{\mathbf{y}} (\mathbf{n} \cdot \nabla)(\mathbf{b} \cdot \nabla) + \tilde{\mathbf{y}}^2 (\mathbf{b} \cdot \nabla)^2] + \kappa \tilde{\mathbf{x}}^2 (\mathbf{n} \cdot \nabla) \right. \right. \\ &\quad \left. \left. + \kappa \tilde{\mathbf{x}} \tilde{\mathbf{y}} (\mathbf{b} \cdot \nabla) + \dots \right) \right\} \frac{\mathbf{t}(s)}{|\mathbf{x} - \mathbf{X}(s)|}. \end{aligned} \quad (12)$$

We shall know from the inner expansion in the following dependence of ζ on φ :

$$\zeta(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \zeta_0 + \zeta_{11} \cos \varphi + \zeta_{12} \sin \varphi + \zeta_{21} \cos 2\varphi + \dots, \quad (14)$$

where

$$\begin{aligned} \zeta_0 &\approx \zeta^{(0)}(r) + \kappa^2 \hat{\zeta}_0^{(2)}(r), & \zeta_{11} &\approx \kappa \hat{\zeta}_{11}^{(1)}(r) + \kappa^3 \zeta_{11}^{(3)}(r), \\ \zeta_{12} &\approx \kappa \hat{\zeta}_{12}^{(1)}(r), & \zeta_{21} &\approx \kappa^2 \hat{\zeta}_{21}^{(2)}(r). \end{aligned} \quad (15)$$

In $\hat{\zeta}_{ij}^{(k)}$, the superscript k stands for order of perturbation, and i labels the Fourier mode with $j = 1$ and 2 being corresponding to $\cos i\theta$ and $\sin i\theta$ respectively.

Substituting (14) and (15) into (13), we get the first two terms \mathbf{A}_m and $\mathbf{A}_{\parallel d}$ as

$$\mathbf{A}_{\parallel}(\mathbf{x}) = \mathbf{A}_m(\mathbf{x}) + \mathbf{A}_{\parallel d}(\mathbf{x}) + \dots, \quad (16)$$

where

$$\begin{aligned} \mathbf{A}_m(\mathbf{x}) &= \frac{\Gamma}{4\pi} \int \frac{\mathbf{t}(s)}{|\mathbf{x} - \mathbf{X}(s)|} ds, \\ \mathbf{A}_{\parallel d}(\mathbf{x}) &= -\frac{1}{16\pi} \left[2\pi \int_0^\infty r^3 \zeta^{(0)} dr \right] \int \frac{\kappa_s \mathbf{n} + \kappa \tau \mathbf{b}}{|\mathbf{x} - \mathbf{X}(s)|} ds \\ &\quad - \frac{d^{(1)}}{2} \int ds [\kappa (\mathbf{n} \cdot \nabla) + \kappa^2] \frac{\mathbf{t}}{|\mathbf{x} - \mathbf{X}(s)|}, \end{aligned} \quad (17)$$

with

$$d^{(1)} = \frac{1}{4\pi} \left\{ \left[2\pi \int_0^\infty r^2 \hat{\zeta}_{11}^{(1)} dr \right] - \frac{1}{2} \left[2\pi \int_0^\infty r^3 \zeta^{(0)} r dr \right] \right\}, \quad (18)$$

being the strength of dipole.

The first term \mathbf{A}_m in (16) pertains to a flow field induced by a curved vortex line of infinitesimal thickness, and is called the “*monopole field*”. The correction term $\mathbf{A}_{\parallel d}$ corresponds to a part of the flow field induced by a *line of dipoles*, based at the vortex centerline, with their axes oriented in the binormal direction. The origin of dipole field is

attributable to the curvature effect; by bending a vortex tube, the vortex lines on the convex side are stretched, while those on the concave side are contracted, producing effectively a vortex pair [6].

The components of vorticity perpendicular to \mathbf{t} make its appearance at $O(\epsilon^2)$.

In view of (5), the second-order terms $\omega_r^{(2)}$ and $\omega_\theta^{(2)}$ are expressible as

$$\omega_r^{(2)} = \frac{\zeta^{(0)}}{v^{(0)}} \hat{\psi}_{11}^{(1)} (\kappa_s \cos\varphi + \kappa\tau \sin\varphi), \quad (19)$$

$$\omega_\theta^{(2)} = \frac{r\zeta^{(0)}}{v^{(0)}} \left[\frac{2}{r} - \frac{\zeta^{(0)}}{v^{(0)}} \hat{\psi}_{11}^{(1)} + \frac{\partial \hat{\psi}_{11}^{(1)}}{\partial r} - rv^{(0)} \right] (\kappa\tau \cos\varphi - \kappa_s \sin\varphi), \quad (20)$$

where $\hat{\psi}_{11}^{(1)}$ will be determined later.

The vector potential \mathbf{A}_\perp associated with the transversal vorticity is, to $O(\epsilon^2)$,

$$\mathbf{A}_\perp(\mathbf{x}) \approx \frac{1}{4\pi} \int \frac{ds}{|\mathbf{x} - \mathbf{X}(s)|} \left[\iint (\omega_r \mathbf{e}_r + \omega_\theta \mathbf{e}_\theta) d\tilde{x} d\tilde{y} \right] \quad (21)$$

Substitution from (19) and (20) yields

$$\mathbf{A}_\perp \approx \frac{1}{4} \left[\int_0^\infty r^2 \hat{\zeta}_{11}^{(1)} dr \right] \int \frac{\kappa_s(s) \mathbf{n}(s) + \kappa(s) \tau(s) \mathbf{b}(s)}{|\mathbf{x} - \mathbf{X}(s)|} ds, \quad (22)$$

the *dipole field* originating from the transversal vorticity. Collecting (17) and (22), we have

$$\mathbf{A} \approx \frac{\Gamma}{4\pi} \int \frac{\mathbf{t}}{|\mathbf{x} - \mathbf{X}|} ds - \frac{d^{(1)}}{2} \int \kappa \mathbf{b} \times \frac{\mathbf{x} - \mathbf{X}}{|\mathbf{x} - \mathbf{X}|^3} ds. \quad (23)$$

The contributions from the monopole field, the first term, and the dipole field the second term, takes tidy forms.

So far only the monopole field has attracted attention, but the dipole field has gone unnoticed. Fukumoto & Okulov (2005) considered the latter for the induced velocity by a helical vortex tube, but its influence on the motion of three-dimensional vortex tube has not been considered, except for the vortex ring and for the localized induction approximation (Fukumoto 2002). In order to gain the vorticity field, we are requested to obtain the inner solution, with the inner limit of the outer solution (23) as the matching condition.

4. Inner solution and motion of a vortex tube

The inner solution is addressed by solving the Euler equations in the moving coordinates. We introduce the dimensionless variables, we write down dimensionless form of the Euler equations and their curl, viewed from the moving coordinates (r, θ, ξ) , along with the subsidiary relation that links ψ to ζ .

The solution at $O(\epsilon)$ is well known (Callegari & Ting 1978). We then make headway to third order.

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